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ON CERTAIN CONTINUED FRACTION REPRESENTATION FOR RATIO OF POLY-BASIC HYPERGEOMETRIC SERIES

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Abstract: In this paper we shall attempt to establish a continued fraction representation for the ratio of two poly-basic series with finite number of parameters each on a different base. In the sequel we also establish a continued fraction representation for a well-poised basic bi-lateral hypergeometric function.

Keywords: Continued fraction, poly-basic hypergeometric series, bilateral hypergeometric series, well-poised.

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1. Introduction

Continued fraction representations for the ratio of two hypergeometric functions occupy prominent place in Ramanujan's contribution to mathematics. In an attempt to prove as well as generalize these results several mathematicians, namely, Bhagirathi [1,2], Bhargava and others [3-5], Denis [6-10] and Singh [14-17] established a large number of results which provide continued fraction representation for the ratio of two ${}_2\phi_1^{\prime s}$ and ${}_3\phi_2^{\prime s}$ of which the former involve general arguments while the later involve constant arguments. There are some scattered results (cf. Denis[10-12]involving bi-basic series with base q and q^2 with general arguments which have their continued fraction representations.

Recently, Denis, Singh and Singh [12] established certain continued fraction representations for the ratio of ${}_{r+1}\phi_r^{\prime s}$ $(3 \leq r \leq 7)$. In the same communication they expressed the ratio of multiple series in terms of continued fraction. In another publication Denis and Singh [11] succeeded in representing ${}_{5k+1}\phi_{5k}$ in terms of k^{th}

$(k \in N)$ power of a continued fraction. All the above transformations have been possible with the help of transformation of basic hypergeometric functions.

In a recent communication Denis et. al [13] gave continued fraction representation for a poly-basic series with four independent bases and pointed out that the same result can be extended to one having any finite number of parameters, each on independent base.

In the present paper we establish certain continued fraction representation for the ratio of two poly-basic hypergeometric series with finite number of independent un-connected bases. In the sequel we shall establish a continued fraction representation for a poly-basic hypergeometric series and deduce a continued fraction representation for a well-poised basic bi-lateral hypergeometric function from this result.

Let us recall the preliminary we shall need in our analysis.

For $|q| < 1$ and α, real or complex we write

$$[\alpha;q]_n \equiv [\alpha]_n = \begin{cases} (1-\alpha)(1-\alpha q)(1-\alpha q^2)...(1-\alpha q^{n-1}), & n \geq 1 \\ 1, \qquad n = 0 \end{cases}$$

Also,

$$\prod[\alpha;q] = \prod[\alpha] = \lim_{n\to\infty}[\alpha]_n = \prod_{n=0}^{\infty}(1-\alpha q^n)$$

Now, we define a basic hypergeometric function (series),

$${}_r\phi_s\left[\begin{array}{l}(a_r);q;z\\(b_s);q^{\lambda}\end{array}\right] \equiv {}_r\phi_s\left[\begin{array}{l}a_1,a_2,...,a_r;q;z\\b_1,b_2,...,b_s;q^{\lambda}\end{array}\right]$$

$$= \sum_{n=0}^{\infty}\frac{[a_1]_n[a_2]_n...[a_r]_n z^n q^{\lambda n(n-1)/2}}{[q;q]_n[b_1]_n[b_2]_n...[b_s]_n}, \tag{1.1}$$

which converges for all $|z| < \infty$ when $\lambda > 0$ and for $|q| < 1$ when $\lambda > 0$.
We defines a basic bilateral hypergeometric function,

$${}_r\psi_r\left[\begin{array}{l}(a_r);q;z\\(b_r);\end{array}\right] \equiv {}_r\psi_r\left[\begin{array}{l}a_1,a_2,...,a_r;q;z\\b_1,b_2,...,b_r\end{array}\right]$$

$$= \sum_{n=-\infty}^{\infty}\frac{[a_1]_n[a_2]_n...[a_r]_n z^n}{[b_1]_n[b_2]_n...[b_r]_n}, \tag{1.2}$$

valid for $|b_1 b_2 ... b_r / a_1 a_2 ... a_r| < |z| < 1$.
We shall define a poly-basic hypergeometric function as,

$$
{}_r\phi_s \left[\begin{array}{l} a_1; p_1 : a_2; p_2 : ... : a_r; p_r; z \\ b_1; q_1 : b_2; q_2 : ... : b_s; q_s \end{array} \right] = \sum_{n=0}^{\infty} \frac{[a_1; p_1]_n [a_2; p_2]_n ... [a_r; p_r]_n z^n}{[b_1; q_1]_n [b_2; q_2]_n ... [b_s; q_s]_n}, \qquad (1.3)
$$

where $p_i (i = 1, 2, ..., r)$ and $q_j (j = 1, 2, ..., s)$ are bases of the parametres $a_i (i = 1, 2, ..., r)$ and $b_j (j = 1, 2, ..., s)$, respectively, and $|z| < 1$.
An expressioon of the type

$$
a_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} ... \frac{a_n}{b_n}
$$

is said to be a terminating continued fraction. It is said to be infinite continued fraction when $n \to \infty$.

2. Main Result

In this section we shall establish our main result. First we prove the following relation,

$$
F[(a_r), (b_s); z] = C[(a_r), (b_s)] F[(a_r p_r), (b_s q_s); z] - H_m[(a_r), (b_s)] \qquad (2.1)
$$

where,

$$
C[(a_r), (b_s)] = \frac{(1 - a_1)(1 - a_2)...(1 - a_r) z}{(1 - b_1)(1 - b_2)...(1 - b_s)}
$$

$$
H_m[(a_r), (b_s)] = \frac{[a_1; p_1]_{m+1} [a_2; p_2]_{m+1} ... [a_r; p_r]_{m+1} z^{m+1}}{[b_1; q_1]_{m+1} [b_2; q_2]_{m+1} ... [b_s; q_s]_{m+1}} - 1
$$

and

$$
F[(a_r), (b_s); z] = \sum_{n=0}^{m} \frac{[a_1; p_1]_n [a_2; p_2]_n ... [a_r; p_r]_n z^n}{[b_1; q_1]_n [b_2; q_2]_n ... [b_s; q_s]_n}.
$$

To prove (2.1) let us consider,

$$
F[(a_r), (b_s); z] - C[(a_r), (b_s)] F[(a_r p_r), (b_s q_s); z] = \sum_{n=0}^{m} \frac{[a_1; p_1]_n [a_2; p_2]_n ... [a_r; p_r]_n z^n}{[b_1; q_1]_n [b_2; q_2]_n ... [b_s; q_s]_n}
$$

$$
- \frac{(1 - a_1)(1 - a_2)...(1 - a_r) z}{(1 - b_1)(1 - b_2)...(1 - b_s)} \times \sum_{n=0}^{m} \frac{[a_1 p_1; p_1]_n [a_2 p_2; p_2]_n ... [a_r p_r; p_r]_n z^n}{[b_1 q_q; q_1]_n [b_2 q_2; q_2]_n ... [b_s q_s; q_s]_n}
$$

$$
= \sum_{n=0}^{m} \frac{[a_1; p_1]_n [a_2; p_2]_n ... [a_r; p_r]_n z^n}{[b_1; q_1]_n [b_2; q_2]_n ... [b_s; q_s]_n} - \sum_{n=0}^{m} \frac{[a_1; p_1]_{n+1} [a_2; p_2]_{n+1} ... [a_r; p_r]_{n+1} z^{n+1}}{[b_1; q_1]_{n+1} [b_2; q_2]_{n+1} ... [b_s; q_s]_{n+1}}
$$

$$= \sum_{n=0}^{m} \frac{[a_1;p_1]_n[a_2;p_2]_n...[a_r;p_r]_n z^n}{[b_1;q_1]_n[b_2;q_2]_n...[b_s;q_s]_n} - \sum_{n=0}^{m-1} \frac{[a_1;p_1]_{n+1}[a_2;p_2]_{n+1}...[a_r;p_r]_{n+1} z^{n+1}}{[b_1;q_1]_{n+1}[b_2;q_2]_{n+1}...[b_s;q_s]_{n+1}}$$

$$- \frac{[a_1;p_1]_{m+1}[a_2;p_2]_{m+1}...[a_r;p_r]_{m+1} z^{m+1}}{[b_1;q_1]_{m+1}[b_2;q_2]_{m+1}...[b_s;q_s]_{m+1}}$$

$$= 1 - \frac{[a_1;p_1]_{m+1}[a_2;p_2]_{m+1}...[a_r;p_r]_{m+1} z^{m+1}}{[b_1;q_1]_{m+1}[b_2;q_2]_{m+1}...[b_s;q_s]_{m+1}},$$

after some simplification. This proves (2.1).

Now, replacing $a_1, a_2, ..., a_r$ and $b_1, b_2, ..., b_s$ by $a_1p_1, a_2p_2, ..., a_rp_r$ and b_1q_1,b_2q_2, ..., b_sq_s, respectively, in(2.1), we get

$$F[(a_rp_r),(b_sq_s);z] = C[(a_rp_r),(b_sq_s)]F[(a_rp_r^2),(b_sq_s^2);z] - H_m[(a_rp_r),(b_sq_s)]. \quad (2.2)$$

Now, (2.1) and (2.2) lead to

$$H_m[(a_rp_r),(b_sq_s)]F[(a_r),(b_s);z] = \{H_m[(a_r),(b_s)] + H_m[(a_rp_r),(b_sq_s)]C[(a_r),(b_s)]\}$$

$$\times F[(a_rp_r),(b_sq_s);z] - H_m[(a_r),(b_s)]C[(a_rp_r),(b_sq_s)]F[(a_rp_r^2),(b_sq_s^2);z]. \quad (2.3)$$

which, in turn leads to

$$\frac{F[(a_r),(b_s);z]}{F[(a_rp_r),(b_sq_s);z]} = A_m[(a_r),(b_s)] - \frac{B_m[(a_r),(b_s)]}{F[(a_rp_r),(b_sq_s);z]/F[(a_rp_r^2),(b_sq_s^2);z]}, \quad (2.4)$$

where

$$A_m[(a_r),(b_s)] = \{H_m[(a_r),(b_s)] + H_m[(a_rp_r),(b_sq_s)]C[(a_r),(b_s)]\}/H_m[(a_rp_r),(b_sq_s)]$$

and

$$B_m[(a_r),(b_s)] = \{H_m[(a_r),(b_s)]C[(a_rp_r),(b_sq_s)]\}/H_m[(a_rp_r),(b_sq_s)].$$

Repeated application of (2.4) yields,

$$\frac{F[(a_r),(b_s);z]}{F[(a_rp_r),(b_sq_s);z]} = A_m[(a_r),(b_s)] - \frac{B_m[(a_r),(b_s)]}{A_m[(a_rp_r),(b_sq_s)]-}$$

$$\frac{B_m[(a_rp_r),(b_sq_s)]}{A_m[(a_rp_r^2),(b_sq_s^2)]-} \frac{B_m[(a_rp_r^2),(b_sq_s^2)]}{A_m[(a_rp_r^3),(b_sq_s^3)] - ...} \quad (2.5)$$

Also, from (2.1) we have

$$F[(a_r),(b_s);z] = -H_m[(a_r),(b_s)]-$$

$$\frac{C[(a_r),(b_s)]H_m[(a_r),(b_s)]}{-C[(a_r),(b_s)]+F[(a_r),(b_s);z]/F[(a_rp_r),(b_sq_s);z]} \tag{2.6}$$

Now, letting $m\to\infty$ in (2.5) we have, after some simplification

$$(1-b_1)(1-b_2)...(1-b_s)\times$$

$$\times\frac{\sum_{n=0}^{\infty}\{[a_1;p_1]_n[a_2;p_2]_n...[a_r;p_r]_nz^n/[b_1;q_1]_n[b_2;q_2]_n...[b_s;q_s]_n\}}{\sum_{n=0}^{\infty}\{[a_1p_1;p_1]_n[a_2p_2;p_2]_n...[a_rp_r;p_r]_nz^n/[b_1q_1;q_1]_n[b_2q_2;q_2]_n...[b_sq_s;q_s]_n\}}$$

$$=(1-b_1)(1-b_2)...(1-b_s)+(1-a_1)(1-a_2)...(1-a_r)z-$$

$$-\frac{(1-a_1p_1)(1-a_2p_2)...(1-a_rp_r)(1-b_1)(1-b_2)...(1-b_s)z}{(1-b_1q_1)(1-b_2q_2)...(1-b_sq_s)+(1-a_1p_1)(1-a_2p_2)...(1-a_rp_r)z-}$$

$$\frac{(1-a_1p_1^2)(1-a_2p_2^2)...(1-a_rp_r^2)(1-b_1q_1)(1-b_2q_2)...(1-b_sq_s)z}{(1-b_1q_1^2)(1-b_2q_2^2)...(1-b_sq_s^2)+(1-a_1p_1^2)(1-a_2p_2^2)...(1-a_rp_r^2)z-}$$

$$\frac{(1-a_1p_1^3)(1-a_2p_2^3)...(1-a_rp_r^3)(1-b_1q_1^2)(1-b_2q_2^2)...(1-b_sq_s^2)z}{(1-b_1q_1^3)(1-b_2q_2^3)...(1-b_sq_s^3)+(1-a_1p_1^3)(1-a_2p_2^3)...(1-a_rp_r^3)z-}$$

$$\frac{(1-a_1p_1^4)(1-a_2p_2^4)...(1-a_rp_r^4)(1-b_1q_1^3)(1-b_2q_2^3)...(1-b_sq_s^3)z}{(1-b_1q_1^4)(1-b_2q_2^4)...(1-b_sq_s^4)+(1-a_1p_1^4)(1-a_2p_2^4)...(1-a_rp_r^4)z-...}. \tag{2.7}$$

This is the continued fraction representation for the ratio of two poly-basic hypergeometric series whose each parameter is on an independent un-connected base.

With $m\to\infty$ in (2.6) and using (2.7), we get the following continued fraction representation of a poly-basic hypergeometric series, as pointed out by Denis et. al [14],

$$\sum_{n=0}^{\infty}\frac{[a_1;p_1]_n[a_2;p_2]_n...[a_r;p_r]_nz^n}{[b_1;q_1]_n[b_2;q_2]_n...[b_s;q_s]_n}$$

$$=1+\frac{(1-a_1)(1-a_2)...(1-a_p)z}{(1-b_1)(1-b_2)...(1-b_s)-\frac{(1-a_1p_1)...(1-a_rp_r)(1-b_1)...(1-b_s)z}{(1-b_1q_1)...(1-b_sq_s)+(1-a_1p_1)...(1-a_rp_r)z}}$$

$$\frac{(1-a_1p_1^2)(1-a_2p_2^2)...(1-a_rp_r^2)(1-b_1q_1)(1-b_2q_2)...(1-b_sq_s)z}{-(1-b_1q_1^2)(1-b_2q_2^2)...(1-b_sq_s^2)+(1-a_1p_1^2)(1-a_2p_2^2)...(1-a_rp_r^2)z-}$$

$$\frac{(1-a_1p_1^3)(1-a_2p_2^3)...(1-a_rp_r^3)(1-b_1q_1^2)(1-b_2q_2^2)...(1-b_sq_s^2)z}{-(1-b_1q_1^3)(1-b_2q_2^3)...(1-b_sq_s^3)+(1-a_1p_1^3)(1-a_2p_2^3)...(1-a_rp_r^3)z-}$$

$$\frac{(1-a_1p_1^4)(1-a_2p_2^4)...(1-a_rp_r^4)(1-b_1q_1^3)(1-b_2q_2^3)...(1-b_sq_s^3)z}{-(1-b_1q_1^4)(1-b_2q_2^4)...(1-b_sq_s^4)+(1-a_1p_1^4)(1-a_2p_2^4)...(1-a_rp_r^4)z-...}. \quad (2.8)$$

This is the continued fraction representation for a poly-basic hypergeometric series whose parameters are on un-connected bases.

Next, we establish the following the following continued fraction representation for a well-poised basic bi-lateral series,

$${}_t\psi_t\left[\begin{array}{l}(a_t);q;q^{(t-1)/2}/a_1a_2...a_t\\ q/(a_t)\end{array}\right]=\sum_{n=-\infty}^{\infty}\frac{[(a_t)]_n q^{(t-1)/2}}{[q/(a_t)]_n(a_1a_2...a_t)^n}$$

$$1+\frac{(1-a_1)(1-a_2)...(1-a_t)(1-q^t)z}{(1-q/a_1)(1-q/a_2)...(1-q/a_t)(1-q)-\frac{A_1}{B_1-}\frac{A_2}{B_2-}\frac{A_3}{B_3-...}}, \quad (2.9)$$

where

$$A_i=(1-a_1q^i)(1-a_2q^i)...(1-a_tq^i)(1-q^i)(1-q^{2i+2})\times$$
$$\times(1-q^i/a_1)(1-q^i/a_2)...(1-q^i/a_t)(1-q^i)(1-q^{2i-2})z$$

and

$$B_i=(1-q^{i+1}/a_1)(1-q^{i+1}/a_2)...(1-q^{i+1}/a_t)(1-q^i)(1-q^{2i})+$$
$$+(1-a_1q^i)(1-a_2q^i)...(1-a_tq^i)(1-q^i)(1-q^{2i+2})z,$$

under suitable convergence conditions.

To prove (2.9), let us take $r=s=t+3$ in (2.8) and then set $p_1=p_2=...=p_t$ $=p_{t+1}=p_{t+2}=p_{t+3}=q, q_1=q_2=....q_t=q_{t+1}=q_{t+2}=q_{t+3}=q$ and $b_i=aq/a_i$, $(i=1,2,..,t)$, $a_{t+1}=a$, $a_{t+2}=\sqrt{a}q, a_{t+3}=\sqrt{a}q$, $b_{t+1}=q$, $b_{t+2}=\sqrt{a}, b_{t+3}=-\sqrt{a}$ and $z=q^{(t-1)/2}/a_1a_2...a_t$ in, the same equation, we get the left hand side, after some simplification

$$\sum_{n=0}^{\infty}\frac{[a_1]_n[a_2]_n...[a_t]_n[a_n][\sqrt{a}q]_n[-\sqrt{a}q]_n q^{(t-1)n/2}}{[q]_n[aq/a_1]_n[aq/a_2]_n...[aq/a_t]_n[\sqrt{a}]_n[-\sqrt{a}]_n(a_1q_2...a_t)^n} \quad (2.10)$$

Now, letting $a\to 1$ in the above it equals

$$1+\sum_{n=1}^{\infty}\frac{[a_1]_n[a_2]_n...[a_t]_n q^{(t-1)n/2}}{[q/a_1]_n[q/a_2]_n...[q/a_t]_n(a_1a_2...a_t)^n}+\sum_{n=1}^{\infty}\frac{[a_1]_n[a_2]_n...[a_t]_n q^{(t+1)n/2}}{[q/a_1]_n[q/a_2]_n...[q/a_t]_n(a_1a_2...a_t)^n}$$

$$=1+\sum_{n=1}^{\infty}\frac{[a_1]_n[a_2]_n...[a_t]_n q^{(t+1)n/2}}{[q/a_1]_n[q/a_2]_n...[q/a_t]_n(a_1a_2...a_t)^n}$$

$$+\sum_{n=-1}^{-\infty}\frac{[a_1]_{-n}[a_2]_{-n}...[a_t]_{-n}q^{(t-1)(-n)/2}}{[q/a_1]_{-n}[q/a_2]_{-n}...[q/a_t]_{-n}(a_1a_2...a_t)^{-n}}$$

$$=1+\sum_{n=1}^{\infty}\frac{[a_1]_n[a_2]_n...[a_t]_nq^{(t-1)n/2}}{[q/a_1]_n[q/a_2]_n...[q/a_t]_n(a_1a_2...a_t)^n}$$

$$+\sum_{n=-1}^{-\infty}\frac{[a_1]_n[a_2]_n...[a_t]_nq^{(t-1)n/2}}{[q/a_1]_n[q/a_2]_n...[q/a_t]_n(a_1a_2...a_t)^n}$$

after some simplifications

$$=\sum_{n=-\infty}^{\infty}\frac{[(a_t)]_nq^{(t-1)n/2}}{[q/(a_t)]_n(a_1a_2...a_t)^n}$$

(2.9) is the continued fraction representation for a well poised basic bilateral hypergeometric function.

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